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REQUEST: A Recursive QUEST Algorithm for Sequential Attitude Determination

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QUEST is a well-known algorithm for least squares fitting of the attitude quaternion of a spacecraft to vector measurements. QUEST, however, is a single time point (single frame) batch algorithm; thus, measurements that were taken at previous time points are discarded. The algorithm presented provides a recursive routine, which considers all past measurements. The algorithm is based on the fact that the so-called K matrix, one of whose eigenvectors is the sought quaternion, is linearly related to the measured vector pairs and on the ability to propagate K. The extraction of the appropriate eigenvector is done according to the classical QUEST algorithm. This stage, however, can be eliminated, and the computation simplified, if a standard eigenvalue-eigenvector solver algorithm is used.

I. Introduction

THE problem of finding attitude from vector observations is stated as follows. A sequence $\boldsymbol{b}_i, i = 1, 2, \ldots, k$, of unit vectors is given. These unit vectors are the result of measurements performed in vehicle Cartesian coordinates of the directions to known objects. The sequence $\boldsymbol{r}_i, i = 1, 2, \ldots, k$, of unit vectors is the sequence of the corresponding unit vectors, resolved in a reference Cartesian coordinate system. We wish to find the attitude matrix A, which transforms vectors from the reference to the body coordinates. Obviously, A has to be an orthogonal matrix. In 1965, Wahba[1] posed this problem as a least squares problem as follows. Let

$$L(A) = \frac{1}{2}\sum_{i=1}^{k} |\boldsymbol{b}_i - A\boldsymbol{r}_i|^2 \tag{1}$$

find the orthogonal 3×3 matrix A that minimizes L. We can weigh each measurement separately according to the accuracy of the particular vector measurement. In addition, we may want to find the quaternion, rather than the matrix, representation of attitude. In such a case, Eq. (1) is replaced by

$$J(\boldsymbol{q}) = \frac{1}{2}\sum_{i=1}^{k} a_i|\boldsymbol{b}_i - A(\boldsymbol{q})\boldsymbol{r}_i|^2 \tag{2}$$

where $a_i, i = 1, 2, \ldots, k$, are the positive weights assigned to each measurement. In Eq. (2) we are looking for that quaternion $\boldsymbol{q}$, which minimizes J. Note that instead of minimizing J, we can maximize g defined as

$$g(\boldsymbol{q}) = 1 - J(\boldsymbol{q}) \bigg/ \left(\frac{1}{2}\sum_{i=1}^{k} a_i\right) \tag{3}$$

It can be shown that $g(\boldsymbol{q})$ can be written as[2]

$$g(\boldsymbol{q}) = \boldsymbol{q}^T K \boldsymbol{q} \tag{4}$$

where K is constructed as follows. Define

$$m_k = \sum_{i=1}^{k} a_i \tag{5a}$$

$$\sigma = \frac{1}{m_k}\sum_{i=1}^{k} a_i \boldsymbol{b}_i^T \boldsymbol{r}_i \tag{5b}$$

$$B = \frac{1}{m_k}\sum_{i=1}^{k} a_i \boldsymbol{b}_i \boldsymbol{r}_i^T \tag{5c}$$

$$S = B + B^T \tag{5d}$$

$$z = \frac{1}{m_k}\sum_{i=1}^{k} a_i(\boldsymbol{b}_i \times \boldsymbol{r}_i) \tag{5e}$$

Then

$$K = \left[\begin{array}{c|c} S - \sigma I & z \\ \hline z^T & \sigma \end{array}\right] \tag{6}$$

where I is the third-order identity matrix. It was shown that $\boldsymbol{q}^*$, of unity length, which maximizes $g(\boldsymbol{q})$ in Eq. (4), satisfies the equation[2]

$$K\boldsymbol{q}^* = \lambda \boldsymbol{q}^* \tag{7}$$

where λ is a yet undetermined Lagrange multiplier. We realize that λ is an eigenvalue of K and $\boldsymbol{q}^*$ is the eigenvector that corresponds to λ. Substitution of this solution into Eq. (4) yields

$$g(\boldsymbol{q}^*) = \lambda \tag{8}$$

and because we wish to maximize g, we choose $\lambda_{\max}$, the largest eigenvalue of K, as the desired eigenvalue, and then $\boldsymbol{q}^*$ is the eigenvector that corresponds to this $\lambda_{\max}$. Once $\lambda_{\max}$ is found, there is no need to solve for the eigenvector of K, because the optimal vector of Rodrigues parameters[3] $\boldsymbol{y}^*$ (also known as Gibbs vector[4]) can be computed as follows:

$$\boldsymbol{y}^* = [(\lambda_{\max} + \sigma)I - S]^{-1}z \tag{9}$$

and the optimal quaternion can be found using the known relation

$$\boldsymbol{q}^* = \frac{1}{\sqrt{1 + |\boldsymbol{y}^*|^2}}\begin{bmatrix} \boldsymbol{y}^* \\ 1 \end{bmatrix} \tag{10}$$

Shuster[5] and Shuster and Oh[6] showed how to easily compute $\lambda_{\max}$ to arbitrary accuracy and how to deal with a singular matrix in Eq. (9). It was also shown there that $\lambda_{\max}$ is close to 1 and is exactly 1 when the measurements are error free.[5,6] [This property is because all a_i in Eq. (2) add up to 1, or equivalently, the introduction of the normalizing factor m_k into Eqs. (5b), (5c), and (5e).] The algorithm for obtaining $\lambda_{\max}$ and $\boldsymbol{q}^*$ from the vector observations discussed is known as the quaternion estimation (QUEST) algorithm.

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QUEST is a robust algorithm. If, for example, we compare it to the extended Kalman filter, we realize that the latter requires the linearization of the nonlinear relations that exist between the measured vectors and the quaternion. This may lead to divergence problems in the use of the extended Kalman filter. QUEST, on the other hand, yields a closed-form solution of the quaternion and thus experiences no divergence problems. QUEST, however, is a single point attitude determination algorithm; that is, it utilizes the vector measurements obtained at a single time point and uses them, and only them, to determine the attitude at that time point. This way, the information contained in past measurements is lost. This has been recognized, and in 1989 Shuster[7] presented an algorithm, which he named filter QUEST, that processes vector measurements recursively. The attitude profile matrix B defined in Eq. (5c), which plays a central role in the algorithm, is updated recursively for use in the QUEST algorithm. Much attention is given in that paper[7] to covariance calculations.

In the present work, the matrix, which is updated recursively, is the K matrix defined in Eqs. (5) and (6). Indeed, as can be seen in the algorithm described earlier, K is the most important element in QUEST. In the following section, we start our presentation of REQUEST by first considering the recursive time-invariant algorithm. Then, in Sec. III, we develop the recursive algorithm for the time-varying case and present an example. In Sec. IV we list the algorithm in a unified form. Finally, in Sec. V, we present our conclusions and recommendation for further work.

II. Recursive Time-Invariant Algorithm

Assume that the body axes are nonrotating with respect to the reference axes. Also assume that k vectors have been processed using the QUEST algorithm. Let K_k denote the K matrix computed by QUEST when processing these k vectors. Now suppose that j new pairs of vectors are measured and we want to use them in the updating of K. The question is then, do we have to recompute the K_{k+j} matrix anew using Eqs. (5) and (6) or can we, perhaps, lump the added j pairs and use them to update K_k? As will be shown next, the latter is possible. In fact, it forms the basis for the REQUEST algorithm. We formulate this quality of K in the following proposition.

Proposition: Let

$$\delta m_{k+j} = \sum_{i=k+1}^{k+j} a_i \tag{11a}$$

where k is the number of, already processed, pairs of vector measurements,

$$\delta\sigma_{k+j} = \sum_{i=k+1}^{k+j} a_i \boldsymbol{b}_i^T \boldsymbol{r}_i \tag{11b}$$

$$\delta B_{k+j} = \sum_{i=k+1}^{k+j} a_i \boldsymbol{b}_i \boldsymbol{r}_i^T \tag{11c}$$

$$\delta S_{k+j} = \delta B_{k+j} + \delta B_{k+j}^T \tag{11d}$$

$$\delta z_{k+j} = \sum_{i=k+1}^{k+j} a_i (\boldsymbol{b}_i \times \boldsymbol{r}_i) \tag{11e}$$

$$\delta K_{k+j} = \left[\begin{array}{c|c} \delta S_{k+j} - \delta\sigma_{k+j} I & \delta z_{k+j} \\ \hline \delta z_{k+j}^T & \delta\sigma_{k+j} \end{array}\right] \tag{11f}$$

then

$$m_{k+j} = m_k + \delta m_{k+j} \tag{11g}$$

and

$$K_{k+j} = (m_k/m_{k+j})K_k + (1/m_{k+j})\delta K_{k+j} \tag{11h}$$

Remark: As mentioned earlier, the computation and use of the coefficient m is done to normalize the weights a_i, such that $\lambda_{\max}$ is close to 1. (See Ref. 6 for the solution of $\lambda_{\max}$.) This, however, is not a necessity because $\lambda_{\max}$ can be found using standard eigenvalue/eigenvector solvers.

Proof: This proposition can be easily proven as follows. Use Eqs. (5) and (6) to compute K_k. Then use this K_k in Eqs. (11) to compute K_{k+j}. It is easy to see that the latter result is identical to the result obtained when all $k+j$ vector pairs are used in Eqs. (5) and (6) to compute K_{k+j} in a single step.

III. Recursive Time-Varying Algorithm

The updating algorithm of the static case can now be extended to the case where the body rotates between measurements. In the ensuing development we distinguish between two cases; namely, the error-free propagation case and the propagation, which is based on angular rate measurement, and as such is contaminated by rate-measurement errors.

A. Error-Free Propagation

Assume that at time t_n, the k pairs were processed, then the body rotated to a new orientation and there, at time t_{n+1}, the j new vector measurements were performed. We wish to find the least squares fit of the quaternion to the first k measurements, at this new time point, and then do the same when the new j measurements are considered, too. Thus, first we are interested in finding $\boldsymbol{q}^*_{n+1/n}$, which is the quaternion that expresses best the attitude at time t_{n+1}, based on the first k measurements, which were performed at time t_n. Let us rewrite the cost function of Eq. (4) for $\boldsymbol{q}$ at time t_n based on the first k measurements, which as mentioned were performed at time t_n,

$$g(\boldsymbol{q}_{n/n}) = \boldsymbol{q}_{n/n}^T K_{n/n} \boldsymbol{q}_{n/n} \tag{12}$$

It is well known (see, e.g., Ref. 4, p. 512) that during the rotation $\boldsymbol{q}$ changes according to the differential equation

$$\dot{\boldsymbol{q}} = \tfrac{1}{2}\Omega \boldsymbol{q} \tag{13}$$

where Ω is a 4×4 skew symmetric matrix whose elements are the body components of the vector of the angular velocity of the body with respect to the reference frame. The solution of Eq. (13) yields

$$\boldsymbol{q}(t_{n+1}) = \Phi(t_{n+1}, t_n)\boldsymbol{q}(t_n) \tag{14}$$

Ideally, when Ω is known perfectly, the matrix $\Phi(t_{n+1}, t_n)$, known as the transition matrix, transforms the quaternion that represents the attitude at time t_n to that which represents attitude at time t_{n+1}. For simplicity of notation, we denote it simply by Φ. The quaternion that we wish to transform from time t_n to time t_{n+1} is $\boldsymbol{q}_{n/n}$; thus we set $\boldsymbol{q}(t_n) = \boldsymbol{q}_{n/n}$. Finally, we denote the quaternion to which $\boldsymbol{q}_{n/n}$ is transformed by $\boldsymbol{q}_{n+1/n}$; thus, we set $\boldsymbol{q}(t_{n+1}) = \boldsymbol{q}_{n+1/n}$. Consequently, Eq. (14) becomes

$$\boldsymbol{q}_{n+1/n} = \Phi \boldsymbol{q}_{n/n} \tag{15}$$

Because Ω is skew symmetric, Φ is orthogonal and we can write

$$\boldsymbol{q}_{n/n} = \Phi^{-1}\boldsymbol{q}_{n+1/n} = \Phi^T \boldsymbol{q}_{n+1/n} \tag{16}$$

Substitution of $\boldsymbol{q}_{n/n}$ of Eq. (16) into Eq. (12) yields

$$g(\boldsymbol{q}_{n/n}) = g'(\boldsymbol{q}_{n+1/n}) = \boldsymbol{q}_{n+1/n}^T \Phi K_{n/n} \Phi^T \boldsymbol{q}_{n+1/n} \tag{17}$$

We realize that the problem of finding $\boldsymbol{q}_{n/n}$ that maximizes g has been transformed into the problem of finding $\boldsymbol{q}_{n+1/n}$ that maximizes g'. Let

$$K_{n+1/n} = \Phi K_{n/n} \Phi^T \tag{18}$$

then Eq. (17) becomes

$$g'(\boldsymbol{q}_{n+1/n}) = \boldsymbol{q}_{n+1/n}^T K_{n+1/n} \boldsymbol{q}_{n+1/n} \tag{19}$$

One may ask whether the problem of finding $\boldsymbol{q}_{n+1/n}$ that maximizes g' is still related to Wahba's problem[1]; that is, will the maximization of g' yield a quaternion that is a least squares fit to the k vector measurements? The answer is yes, because maximizing $\boldsymbol{q}_{n+1/n}$ is directly related, through Eq. (16), to $q_{n/n}$, which maximizes Eq. (12) and is the solution of Wahba's problem, given the k measurements. When adjoining the constraint $|q_{n+1/n}|^2 = 1$ (using a Lagrange multiplier $\lambda_{n+1/n}$) to the $g'(q_{n+1/n})$ function of Eq. (19) and using the first-order necessary condition for a maximum of the adjoined function, it can be shown that as before, $q^*_{n+1/n}$, which is the maximizing $q_{n+1/n}$, satisfies the equation

$$K_{n+1/n} q^*_{n+1/n} = \lambda_{n+1/n} q^*_{n+1/n} \tag{20}$$

and that $q^*_{n+1/n}$ is the eigenvector of $K_{n+1/n}$ that corresponds to the largest eigenvalue of $K_{n+1/n}$. Note that although we assume error-free propagation, the measured vectors contain measurement errors. Finally, note that $K_{n+1/n}$ being a result of a similarity transformation on $K_{n/n}$ has the eigenvalues of the latter even though its eigenvectors are different.

Now that we have established that $K_{n+1/n}$ is the proper K matrix for finding the least squares fit of the quaternion at time t_{n+1} based on all past k measurements, we can include additional j measurements performed at t_{n+1}. For this we use Eq. (11h) of the Proposition. Consequently, from Eqs. (18) and (11h) we obtain

$$K_{n+1/n} = \Phi K_{n/n} \Phi^T \tag{21a}$$

$$K_{n+1/n+1} = (m_n/m_{n+1}) K_{n+1/n} + (1/m_{n+1}) \delta K_{n+1} \tag{21b}$$

where $m_n = m_k$, and where m_{n+1} is computed according to Eq. (11g) and δK_{n+1} according to Eq. (11f). We demonstrate the algorithm by the following example.

Example: Given are four error-free vectors in the reference coordinate frame:

$$r1 = \begin{bmatrix} 0.267 \\ 0.535 \\ 0.802 \end{bmatrix} \qquad r2 = \begin{bmatrix} -0.667 \\ -0.667 \\ -0.333 \end{bmatrix}$$

$$r3 = \begin{bmatrix} 0.267 \\ -0.802 \\ 0.535 \end{bmatrix} \qquad r4 = \begin{bmatrix} -0.447 \\ 0.894 \\ 0.000 \end{bmatrix}$$

and a rotation from the reference to body axes described by the following Euler vector:

$$\phi^T = [0.9, 0.2, 0.8]$$

The corresponding quaternion is

$$q(1)^T = [0.423, 0.094, 0.376, 0.819]$$

The four r vectors are transformed to the body frame, and noise is added to the transformed vectors. The noise elements added to each component of the transformed vectors is drawn from a random number generator. The standard deviation of the noises are

$$\sigma 1 = 0.01 \qquad \sigma 2 = 0.05 \qquad \sigma 3 = 0.03 \qquad \sigma 4 = 0.02$$

The vectors are then normalized. The resulting simulated measured vectors in the body frame are then

$$b1 = \begin{bmatrix} 0.688 \\ 0.662 \\ 0.297 \end{bmatrix} \qquad b2 = \begin{bmatrix} -0.985 \\ -0.120 \\ -0.123 \end{bmatrix}$$

$$b3 = \begin{bmatrix} -0.280 \\ -0.030 \\ 0.959 \end{bmatrix} \qquad b4 = \begin{bmatrix} 0.303 \\ 0.575 \\ -0.760 \end{bmatrix}$$

and the weights are chosen to be

$$a_i = \sigma_i^{-2} \qquad i = 1, 2, 3, 4$$

1. Application of QUEST to the First Two Pairs

Using initially, at Time t_1, the first two pairs of vectors, $r1$ and $b1$ and $r2$ and $b2$, we obtain $K_{1/1}$. Its largest eigenvalue and the corresponding eigenvector, which according to our notation is $q_{1/1}$, are

$$\lambda_{1/1} = 1.0003551 \qquad q^T_{1/1} = [0.427, 0.105, 0.383, 0.813]$$

[Note that from Eqs. (2), (3), and (8), λ_{max} has to be less than 1. We realize, however, that $\lambda_{1/1}$ is slightly larger than 1. This discrepancy stems from the fact that $\lambda_{1/1}$ was computed using an eigenvalue/eigenvector solver and not using Eq. (3).] The corresponding estimated transformation matrix $A_{1/1}$, the true matrix $A(1)$, and the difference (error) matrix are

$$A_{1/1} = \begin{bmatrix} 0.685 & 0.712 & 0.156 \\ -0.532 & 0.343 & 0.774 \\ 0.497 & -0.613 & 0.614 \end{bmatrix}$$

$$A(1) = \begin{bmatrix} 0.700 & 0.695 & 0.164 \\ -0.536 & 0.361 & 0.763 \\ 0.471 & -0.622 & 0.625 \end{bmatrix}$$

$$A_{1/1} - A(1) = \begin{bmatrix} -0.015 & 0.017 & -0.007 \\ 0.004 & -0.018 & 0.011 \\ 0.026 & 0.009 & -0.012 \end{bmatrix}$$

The Euclidean norm of the error matrix is

$$|A_{1/1} - A(1)| = 0.044 \tag{22}$$

This error stems, of course, from the measurement error contained in the b vectors.

2. Rotation of the Body Coordinate System

We assume that after processing the first two pairs, which yielded $A_{1/1}$, the body rotates for 1 s at the following angular rate (in radian per second):

$$\omega^T = [0.1, 0.2, -0.3]$$

The matrix Φ, which propagates the quaternion of this rotation [see Eq. (15)], and the attitude matrix ΔA, which expresses the change in the body coordinates, are

$$\Phi = \begin{bmatrix} 0.983 & -0.149 & -0.099 & 0.050 \\ 0.149 & 0.983 & 0.050 & 0.099 \\ 0.099 & -0.050 & 0.983 & -0.149 \\ -0.050 & -0.099 & 0.149 & 0.983 \end{bmatrix}$$

$$\Delta A = \begin{bmatrix} 0.936 & -0.283 & -0.210 \\ 0.303 & 0.951 & 0.068 \\ 0.181 & -0.127 & 0.975 \end{bmatrix}$$

3. Measurement Update of K

We use ΔA to transform b_3 and b_4 to the new time point t_2. Using these b as the simulated measurements at t_2, we compute δK_2 according to Eq. (11), and update K, using Eq. (21), as follows:

$$K_{2/1} = \Phi K_{1/1} \Phi^T$$

$$K_{2/2} = (m_1/m_2) K_{2/1} + (1/m_2) \delta K_2$$

The largest eigenvalue and the corresponding eigenvector of $K_{2/2}$ are

$$\lambda_{2/2} = 1.0001957 \qquad q^T_{2/2} = [0.402, 0.253, 0.282, 0.834]$$

The corresponding estimated attitude matrix is

$$A_{2/2} = \begin{bmatrix} 0.713 & 0.673 & -0.195 \\ -0.267 & 0.518 & 0.813 \\ 0.648 & -0.528 & 0.549 \end{bmatrix}$$

4. Check

We now check this result as follows. We use ΔA to transform $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ to the new time point t_2. (Recall that $\boldsymbol{b}_3$ and $\boldsymbol{b}_4$ were already transformed to compute δK_2.) Now we apply the QUEST algorithm to all four pairs of $\boldsymbol{r}$ and $\boldsymbol{b}$. The resulting quaternion should be equal to the quaternion updated by the REQUEST algorithm. Indeed the two quaternions agree to at least 10^{-12}.

Remark: When we compare $A(2) = \Delta A \cdot A(1)$, which is the correct matrix that transforms from the reference to body axes at t_2, to the attitude matrix $A_{2/2}$, obtained by REQUEST (and by QUEST as well), we obtain

$$A_{2/2} - A(2) = \begin{bmatrix} 0.005 & -0.006 & 0.000 \\ -0.001 & 0.007 & -0.005 \\ -0.006 & -0.001 & 0.007 \end{bmatrix}$$

The Euclidean norm of this error matrix is

$$|A_{2/2} - A(2)| = 0.015$$

This error stems from the measurement noise in the $\boldsymbol{b}$ vectors and not from the algorithm. Note that the latter error is smaller than $|A_{1/1} - A(1)|$ shown in Eq. (22). This is expected, since $A_{2/2}$ is computed using four pairs of vectors, whereas $A_{1/1}$ is computed using only two.

B. Noisy Propagation

In the preceding developments we considered the presence of noise only in the measurements and assumed that the angular rate vector ω, was known to us perfectly. We wish to consider now errors also in our knowledge of ω. Let us denote the measured, or computed, ω by ω_m. We also assume that the error is additive; thus we can write

$$\omega_m = \omega + \varepsilon \tag{23}$$

where ε is the error component in the measured angular rate vector. We distinguish between two cases; namely, short time application and long time application of REQUEST.

1. Short Mission Duration

Since a typical update rate is once per second, typical gyro noise does not cause a considerable attitude error during such a short period. In fact, even with an update rate of once per 10 s, the attitude error is very small. To illustrate this point, we turn to the example. Suppose that we use a triad of single axis gyros, each having a constant drift rate of 1 deg/h, which is about 100 times larger than that of inertial grade gyros. We use the first three measurements to compute $K_{1/1}$ and from it, $A_{1/1}$. We then propagate $K_{1/1}$ using Φ_m, the gyro-error ridden transition matrix, and obtain $K_{2/1,m}$, and then compute the corresponding attitude matrix $A_{2/1,m}$. In parallel we do the same using Φ, the correct transition matrix, and obtain $A_{2/1}$, the corresponding attitude matrix. Doing so, we discover that the largest difference between the magnitude of the elements of the two attitude matrices, $A_{2/1,m}$ and $A_{2/1}$, is less than 5.23×10^{-6}. Next, following the REQUEST algorithm, we use the fourth measurement at time t_2 to compute δK_2, update both $K_{2/1,m}$ and $K_{2/1}$, and compute the corresponding attitude matrix for the correct and erroneous propagations. The largest error between the elements of the two updated attitude matrices is less than 2.55×10^{-6}. We see two interesting facts. First, the gyro error has little effect on the propagated K and, consequently, on the propagated and updated attitude matrices. Second, the incorporation of a new measurement further reduces the small error caused by gyro drift. As a consequence, we conclude that for a short mission duration the buildup of attitude errors due to gyro drift is negligible and the algorithm given in Eqs. (21) is adequate.

2. Long Mission Duration

Space missions where QUEST is traditionally being used are of long duration; therefore, the initial measurements are propagated through the repeated use of Eq. (21a) to the current time. This in turn reduces the accuracy of those measurements, and as time goes by they may corrupt the attitude rather than improve it. Consequently, we wish to gradually reduce the influence of old measurements, and eventually eliminate them altogether. This is usually done using the fading memory concept.[8] Accordingly, instead of using Eq. (21b) for updating K, we may want to use the following algorithm:

$$K_{n+1/n+1} = \rho_n(m_n/m_{n+1})(K_{n+1/n} + 1/m_{n+1})\delta K_{n+1} \tag{24a}$$

where $0 < \rho_n \leq 1$. Note that ρ_n has to be larger than 0 for Eq. (24a) to yield a meaningful K when only one measurement is performed at t_{n+1}. Also note that when no process noise is present we set $\rho_n = 1$, which keeps the same relative weighting of past and present measurements as in Eq. (21b). The value of ρ_n can be determined experimentally where a larger propagation noise is compensated by a smaller ρ_n value. Note that ρ_n can vary from step to step, allowing the consideration of changing gyro noise statistics. Note that the introduction of m_n in the REQUEST algorithm stems from our wish to maintain $\lambda_{\max} \cong 1$. This is important if we use the classical QUEST method for solving for $\lambda_{\max}$ (see Ref. 6). (If we use a given eigenvalue/eigenvector solver routine, this is irrelevant.)

Another approach to the introduction of the weighting factor ρ_n, which is consistent with the logic behind the development of the recursive time-invariant algorithm, is as follows. At each time point we compute $m_{n+1} = m_n + \delta m_{n+1}$. The weights m_n and m_{n+1} are used to weigh properly the old and the new information. When the old K is propagated using noisy information, we want to give the propagated K a smaller weight than in the ideal case; therefore, we replace the weight m_n by $\rho_n m_n$. Consequently, the updating formula for m becomes $m_{n+1} = \rho_n m_n + \delta m_{n+1}$. Using these weightings in Eq. (21b) yields

$$K_{n+1/n+1} = \frac{\rho_n m_n}{\rho_n m_n + \delta m_{n+1}} K_{n+1/n} + \frac{1}{\rho_n m_n + \delta m_{n+1}} \delta K_{n+1} \tag{24b}$$

Note that this updating formula keeps $\lambda_{\max}$ close to 1.

IV. Algorithm Summary

The REQUEST algorithm is summarized as follows.

1) Use the k measurements performed at the starting point t_1 to compute $K_{1/1}$ according to Eqs. (5) and (6).

2) Form the angular rate matrix

$$\frac{1}{2}\Omega = \frac{1}{2}\begin{bmatrix} 0 & \omega_z & -\omega_y & \omega_x \\ -\omega_z & 0 & \omega_x & \omega_y \\ \omega_y & -\omega_x & 0 & \omega_z \\ -\omega_x & -\omega_y & -\omega_z & 0 \end{bmatrix} \tag{25}$$

where ω_i, $i = 1, 2, 3$, are the components of the body axes angular rate vector.

3) Compute Φ, the transition matrix from time t_1 to time t_2, corresponding to this angular rate matrix. [Algorithms for computing Φ can be found in standard control theory or state estimation texts (see, e.g., Ref. 9).]

4) Propagate $K_{1/1}$ according to

$$K_{2/1} = \Phi K_{1/1} \Phi^T \tag{26}$$

5) Compute δK_2 as follows:

$$\delta m_2 = \sum_{i=k+1}^{k+j} a_i \tag{27a}$$

(where k is the number of pairs of vector measurements already processed and j is the number of new measurement pairs performed at time t_2),

$$\delta\sigma_2 = \sum_{i=k+1}^{k+j} a_i \boldsymbol{b}_i^T \boldsymbol{r}_i \tag{27b}$$

$$\delta B_2 = \sum_{i=k+1}^{k+j} a_i \boldsymbol{b}_i \boldsymbol{r}_i^T \tag{27c}$$

$$\delta S_2 = \delta B_2 + \delta B_2^T \tag{27d}$$

$$\delta z_2 = \sum_{i=k+1}^{k+j} a_i (\boldsymbol{b}_i \times \boldsymbol{r}_i) \tag{27e}$$

$$\delta K_2 = \left[\begin{array}{c|c} \delta S_2 - \delta\sigma_2 I & \delta z_2 \\ \hline \delta z_2^T & \delta\sigma_2 \end{array}\right] \tag{27f}$$

then set ρ_1 in the range

$$0 < \rho_1 \leq 1 \tag{27g}$$

and compute

$$K_{2/2} = \frac{\rho_1 m_1}{\rho_1 m_1 + \delta m_2} K_{2/1} + \frac{1}{\rho_1 m_1 + \delta m_2} \delta K_2 \tag{27h}$$

In preparation for the next time update, compute

$$m_2 = m_1 + \delta m_2 \tag{27i}$$

6) Only if there is an interest in extracting the attitude from $K_{2/2}$, compute the attitude at this time point (t_2); otherwise go to step 7. The extraction of attitude from $K_{2/2}$ can be done using the algorithm given in QUEST, or any standard software package that can compute eigenvalues and eigenvectors of a real symmetric matrix. If the latter approach is chosen, then select the largest eigenvalue of $K_{2/2}$ and compute the corresponding eigenvector.

7) Go to step 2 and increase the appropriate indices by 1, or stop if so desired.

V. Conclusions and Recommendations

We presented a recursive algorithm for attitude determination from vector observations that was derived from QUEST. The new recursive algorithm, which we call REQUEST, is based on the propagation and update of the K matrix, one of whose eigenvectors is the sought attitude quaternion. Using REQUEST, we do not lose information gathered by measurements performed at previous time points, and because we use prior information, even one measurement at a particular time point to which K is propagated is sufficient for updating the attitude. We showed how to apply the algorithm to cases where more than one measurement is taken at the new time point. We demonstrated that under normal conditions, and for short mission durations, there is no need to treat propagation noise (also known as process noise). For long mission durations, we do have to consider the process noise. This is done using the fading memory notion whereby the weight of the contribution of old measurements to K is reduced with time. We presented an example to illustrate the algorithm.

As mentioned, the new algorithm shows how to propagate and update K, but once K is computed its largest eigenvalue and the corresponding eigenvector, which is the sought quaternion, are found using the method of QUEST. If, however, a standard eigenvalue/eigenvector solver is used, then the eigenvalue and eigenvector can be found directly without solving for Rodrigues parameters, and without the need to be concerned about matrix singularity problems [see Eq. (9)].

As a follow up to this work, it is recommended that REQUEST be tested using real spacecraft data and be tested against other recursive algorithms such as the extended Kalman filter. As mentioned in the Introduction, QUEST is a single point attitude determination algorithm; therefore, it is incapable of estimating gyro drift. REQUEST, however, yields quaternion estimates, which are function of gyro bias. Therefore, it is logical to try to expand REQUEST to include gyro bias estimates. It is recommended then that the addition of bias estimation capability to REQUEST be investigated.

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